Waves in Multicomponent Vlasov Plasmas with Anisotropic Pressure

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This is a study of the dispersion formulas for small amplitude waves in a fully ionized N-component plasma, in the presence of a constant external magnetic field. The number of ion species (whether positively or negatively charged) is left general. From a BOLTZMANN-VLASOV equation for each component of the plasma the first three moment equations are taken. The low-temperature approximation is used to close the set of equations. This set is then solved together with the equations of MAXWELL to obtain a general dispersion relation, a determinant of order $3N$. This relation is studied for the principal waves, and various compact formulas are derived. They are shown to include several known results, when applied to plasmas of the usual compositions. Their general form makes them suitable for various physical approximations.

1. Introduction

The propagation of free linear waves through plasmas has received an enormous attention these last ten years (e. g. ALLIS [1], DENISSE-DELCROIX [2]). There has been little or no change, however, in the composition of the plasmas under consideration. A plasma is almost invariably assumed to consist of electrons and of one species of ions only, preferably singly charged. From time to time this basic model is modified to a partly ionized plasma, which includes also one sort of neutrals.

Obviously, this approach is directly inspired by the most promising laboratory plasma, the hydrogen one. A treatment of a plasma as an assembly of particles belonging to N different species seems but of academic interest and can only be found in some books (STIX [3], BRANDSTATTER [4]). At times authors most cheerfully start with an N-component plasma, but narrow their views after some pages to the usual two-component case. We use the expression component of a plasma to designate the collection of all like particles in a plasma.

Nevertheless, there may appear phenomena, such as the ion-ion hybrid resonance in a plasma with two different species of ions [5], which require the extension of the treatment of plasma waves beyond the limitations occurring in the usual compositions reviewed. It is our purpose to make in our calculations no assumptions whatever as to the actual composition of the plasma, in order to derive generally valid dispersion relations.

As a model of the plasma we will take an infinite VLASOV plasma. The use of the full BOLTZMANN equations is impossibly difficult, and hence in this first approximation theory every interaction is left out of account. We also will not trouble ourselves here with boundary conditions.

The wave phenomena studied here will be interpreted as perturbations of the stationary state of the plasma. The difference between the perturbed and the equilibrium value of every variable will be regarded as a small quantity. As to external forces, we admit but a constant magnetic field.

To start with, we derive from a BOLTZMANN-VLASOV equation for each constituent of a plasma in the presence of an electromagnetic field the first three moment equations. The inclusion of the third moment equation allows us to avoid the rather unrealistic condition of a separate adiabatic behaviour of the plasma components, and admits an anisotropic pressure, both in the equilibrium and in the perturbed state. After the now well-known linearization of these equations, as well as of the equations of MAXWELL, we eliminate the first-order electromagnetic field. This yields a set of $3N$ equations in the $3N$ unknowns, the components of the N first-order drift velocities. The dispersion relation follows immediately, if we wish a nontrivial solution for these drift velocities. The discussion of

[1] W. P. ALLIS, S. J. BUCHSBAUM, and A. BERS, Waves in Anisotropic Plasmas, M.I.T. Press, Cambridge, Mass. 1963.
[2] J. F. DENISSE and J. L. DELCROIX, Théorie des ondes dans les plasmas, Dunod, Paris 1961.
[3] T. H. STIX, The Theory of Plasma Waves, McGraw-Hill Book Co., New York 1962.
[4] J. J. BRANDSTATTER, An Introduction to Waves, Rays and Radiation in Plasma Media, Mc Graw-Hill Book Co., New York 1963.
[5] S. J. BUCHSBAUM, Phys. Fluids **3**, 418 [1960].

this dispersion equation for different orientations of the external magnetic field shows the possibility of finding compact formulas in some limited cases.

2. Basic Equations

The particle distribution function f_a for each species a of a plasma obeys in the presence of an electric field $\boldsymbol{E}$ and of a magnetic induction $\boldsymbol{B}$ a BOLTZMANN-VLASOV equation:

$$\frac{\partial f_a}{\partial t} + \boldsymbol{w}_a \bullet \nabla f_a + \frac{q_a}{m_a} (\boldsymbol{E} + \boldsymbol{w}_a \times \boldsymbol{B}) \bullet \nabla_w f_a = 0 . \quad (2.1)$$

Here $\boldsymbol{w}_a$, q_a and m_a respectively represent the individual velocity, charge and mass of a particle of kind a. t stands for the time and ∇, ∇_{w_a} are the position and velocity gradient operators.

Throughout this paper we will use the indices a and b to designate the particle species. They run from 1 to N, if N denotes the number of components of the plasma. No neutral particle species are considered, as in a VLASOV plasma their motion is not coupled to the motion of the charged particles. They hence are left out of this macroscopic picture. Thus (2.1) is a shorthand notation to avoid the explicit writing, at every step, of a set of N similar equations. It is also understood that each subsequent summation or product over a or b is from 1 to N, except where else stated.

We now introduce the customary definitions of respectively the density n_a, the mean drift velocity $\boldsymbol{v}_a$, the pressure tensor $\mathbf{P}_a$ and the heat flow tensor $\mathbf{Q}_a$ for each species of the plasma particles as

$$n_a = \int f_a \, d^3 \boldsymbol{w}_a , \quad (2.2)$$

$$n_a \boldsymbol{v}_a = \int \boldsymbol{w}_a f_a \, d^3 \boldsymbol{w}_a , \quad (2.3)$$

$$\mathbf{P}_a = m_a \int (\boldsymbol{w}_a - \boldsymbol{v}_a)(\boldsymbol{w}_a - \boldsymbol{v}_a) f_a \, d^3\boldsymbol{w}_a , \quad (2.4)$$

$$\mathbf{Q}_a = m_a \int (\boldsymbol{w}_a - \boldsymbol{v}_a)(\boldsymbol{w}_a - \boldsymbol{v}_a)(\boldsymbol{w}_a - \boldsymbol{v}_a) f_a \, d^3\boldsymbol{w}_a . \quad (2.5)$$

After multiplying (2.1) respectively with 1, $m_a \boldsymbol{w}_a$ and $m_a(\boldsymbol{w}_a - \boldsymbol{v}_a)(\boldsymbol{w}_a - \boldsymbol{v}_a)$ and integrating over the whole velocity space, we find the first three moment equations

$$\frac{\partial n_a}{\partial t} + \nabla \bullet (n_a \boldsymbol{v}_a) = 0 , \quad (2.6)$$

$$\frac{\partial \boldsymbol{v}_a}{\partial t} + (\boldsymbol{v}_a \bullet \nabla) \boldsymbol{v}_a + \frac{1}{\varrho_a} \nabla \bullet \mathbf{P}_a - \frac{q_a}{m_a} (\boldsymbol{E} + \boldsymbol{v}_a \times \boldsymbol{B}) = 0 , \quad (2.7)$$

$$\frac{\partial \mathbf{P}_a}{\partial t} + \nabla \bullet (\mathbf{Q}_a + \boldsymbol{v}_a \mathbf{P}_a) + \mathbf{P}_a \bullet \nabla \boldsymbol{v}_a + (\mathbf{P}_a \bullet \nabla \boldsymbol{v}_a)^{\mathrm{T}} + \frac{q_a}{m_a} (\boldsymbol{B} \times \mathbf{P}_a + (\boldsymbol{B} \times \mathbf{P}_a)^{\mathrm{T}}) = 0 . \quad (2.8)$$

In (2.7) ϱ_a is the mass density of the constituent a, whereas in (2.8) the superscript T denotes the transposed tensor. We also will need the equations of MAXWELL

$$\nabla \times \boldsymbol{E} = - \partial \boldsymbol{B} / \partial t , \quad (2.9)$$

$$\nabla \times \boldsymbol{B} = \frac{1}{c^2} \frac{\partial \boldsymbol{E}}{\partial t} + \mu_0 \boldsymbol{j} , \quad (2.10)$$

$$\nabla \bullet \boldsymbol{B} = 0 . \quad (2.11)$$

The current density $\boldsymbol{j}$ is given by

$$\boldsymbol{j} = \sum_a n_a q_a \boldsymbol{v}_a . \quad (2.12)$$

3. Linearization

We wish to focus our attention to the small amplitude waves, as perturbations about the equilibrium state of the plasma. We assume it therefore possible to split every variable into two parts. The first part or zero-order part gives the equilibrium value of that variable, supposed to be constant and written with a capital letter. The other or first-order part gives the fluctuations of the variable and is written in a plane wave solution form, with its amplitude represented by the corresponding lower case letter. We henceforth will neglect squares, higher powers and products of these small amplitudes. Any quantity x_a or X_a thus is replaced by

$$X_a + x_a \exp(i \, \boldsymbol{k} \cdot \boldsymbol{r} - i \, \omega t) \quad (3.1)$$

where $\boldsymbol{k}$ represents the wavevector and ω the angular frequency of the proposed plane wave solution.

The electromagnetic field consists of a constant external magnetic field in addition to the first-order electromagnetic fields associated with the wave. We direct the z-axis of our coordinate system along the external magnetic induction and rotate the system around the z-axis so that the x, z-plane contains the wavevector. This may be done without restricting the obtained solutions in generality.

Some further remarks must be made here. First of all, no net drift velocities are considered in the steady state, although this does not preclude any thermal motions. Secondly, the equilibrium pressure tensors P_a may be represented in the above defined reference frame as

$$\mathsf{P}_a = P_\perp^a \, \mathsf{I} + (P_\parallel^a - P_\perp^a)\, \boldsymbol{z}\boldsymbol{z} \tag{3.2}$$

where I is a unit tensor and $\boldsymbol{z}$ represents a unit vector along the z-axis.

Finally, we wrote nothing about the heat flow tensor, as we will work in the so-called low-temperature or fully adiabatic case[6], where the phase velocity of the wave is much greater than any thermal velocity of the plasma components. It then is legitimate to neglect completely $\nabla \cdot \mathsf{q}_a$. Substitution of the solutions (3.1) into (2.6—12) yields the linearized form of these equations:

$$\omega\, n_a = N_a\, \boldsymbol{k} \cdot \boldsymbol{v}_a\,, \tag{3.3}$$

$$\omega\, \boldsymbol{v}_a - \frac{1}{\varrho_a} \boldsymbol{k} \cdot \mathsf{p}_a - i\, \frac{q_a}{m_a} (\boldsymbol{e} + \boldsymbol{v}_a \times B \hat{\boldsymbol{z}}) = 0\,, \tag{3.4}$$

$$\begin{aligned} \omega\, \mathsf{p}_a &- \mathsf{P}_a(\boldsymbol{k} \cdot \boldsymbol{v}_a) - \mathsf{P}_a \cdot \boldsymbol{k} \boldsymbol{v}_a - (\mathsf{P}_a \cdot \boldsymbol{k} \boldsymbol{v}_a)^{\mathrm{T}} \\ &+ i\, \frac{q_a}{m_a}\, B[\hat{\boldsymbol{z}} \times \mathsf{p}_a + (\hat{\boldsymbol{z}} \times \mathsf{p}_a)^{\mathrm{T}}] \\ &+ i\, \frac{q_a}{m_a} [\boldsymbol{b} \times \mathsf{P}_a + (\boldsymbol{b} \times \mathsf{P}_a)^{\mathrm{T}}] = 0\,, \end{aligned} \tag{3.5}$$

$$\boldsymbol{k} \times \boldsymbol{e} = \omega\, \boldsymbol{b}\,, \tag{3.6}$$

$$\boldsymbol{k} \times \boldsymbol{b} = -(\omega/c^2)\, \boldsymbol{e} - i\, \mu_0 \boldsymbol{j}\,, \tag{3.7}$$

$$\boldsymbol{k} \cdot \boldsymbol{b} = 0\,, \tag{3.8}$$

$$\boldsymbol{j} = \sum_a N_a q_a \boldsymbol{v}_a\,. \tag{3.9}$$

Due to the special choice of B along the z-axis and of P_a as given by (3.2), we obtained

$$\hat{\boldsymbol{z}} \times \mathsf{P}_a + (\hat{\boldsymbol{z}} \times \mathsf{P}_a)^{\mathrm{T}} = 0\,.$$

In all these equations the position and time dependence factor has been suppressed. (3.3) merely serves to express n_a as a function of the drift velocities and will not be used anymore.

4. Electromagnetic Fields

The elimination of $\boldsymbol{b}$ between (3.6) and (3.7), by taking the cross product of (3.6) with $\boldsymbol{k}$, yields an equation in $\boldsymbol{e}$:

$$\boldsymbol{k} \times (\boldsymbol{k} \times \boldsymbol{e}) = -(\omega^2/c^2)\, \boldsymbol{e} - i\, \mu_0\, \omega \boldsymbol{j}$$

or

$$(\boldsymbol{k} \cdot \boldsymbol{e})\, \boldsymbol{k} + (\omega^2/c^2 - k^2)\, \boldsymbol{e} = -i\, \mu_0\, \omega \boldsymbol{j}\,.$$

Dotting (3.7) with $\boldsymbol{k}$ gives

$$\boldsymbol{k} \cdot \boldsymbol{e} = \boldsymbol{k} \cdot \boldsymbol{j} / i\, \varepsilon_0\, \omega\,.$$

Putting for shortness

$$\alpha = \omega^2/c^2 - k^2 \tag{4.1}$$

we obtain for $\boldsymbol{e}$ the following expression

$$\boldsymbol{e} = i\, \boldsymbol{k}(\boldsymbol{k} \cdot \boldsymbol{j}) / \alpha\, \varepsilon_0\, \omega - i(\omega\, \mu_0/\alpha)\, \boldsymbol{j}\,. \tag{4.2}$$

From (3.9) there follows that

$$\boldsymbol{e} = (i/\varepsilon_0\, \alpha\, \omega) \sum_b N_b q_b [\boldsymbol{k}(\boldsymbol{k} \cdot \boldsymbol{v}_b) - (\omega^2/c^2)\, \boldsymbol{v}_b]\,.$$

The definition of the partial plasma angular frequency Π_a, given by

$$\Pi_a^2 = N_a q_a{}^2 / \varepsilon_0\, m_a \tag{4.3}$$

and of the partial charge density

$$\sigma_a = N_a q_a \tag{4.4}$$

allow us to put

$$q_a / \varepsilon_0\, m_a = \Pi_a^2 / \sigma_a\,.$$

With the help of these definitions we achieve the desired form for the first-order electric field as

$$\frac{q_a}{m_a}\, \boldsymbol{e} = \frac{i}{\alpha\, \omega} \frac{\Pi_a^2}{\sigma_a} \sum_b \sigma_b \left(\boldsymbol{k}(\boldsymbol{k} \cdot \boldsymbol{v}_b) - \frac{\omega^2}{c^2}\, \boldsymbol{v}_b \right). \tag{4.5}$$

In a similar way we may obtain the equation governing $\boldsymbol{b}$ if we take the cross product of (3.7) with $\boldsymbol{k}$ and eliminate $\boldsymbol{e}$ between this new equation and (3.6). It yields

$$\boldsymbol{k} \times (\boldsymbol{k} \times \boldsymbol{b}) = -(\omega^2/c^2)\, \boldsymbol{b} - i\, \mu_0\, \boldsymbol{k} \times \boldsymbol{j}$$

or according to (3.8)

$$\boldsymbol{b} = -(i/\alpha)\, \mu_0\, \boldsymbol{k} \times \boldsymbol{j}\,. \tag{4.6}$$

The current density remains given by (3.9) and hence we find in an analogous manner as for the electric field the expression for the first-order magnetic induction

$$(q_a/m_a)\, \boldsymbol{b} = -(i/\alpha\, c^2)\, (\Pi_a^2/\sigma_a) \sum_b \sigma_b\, \boldsymbol{k} \times \boldsymbol{v}_b\,. \tag{4.7}$$

[6] I. B. Bernstein and S. K. Trehan, Nucl. Fusion 1, 3 [1960].

5. Pressure Tensors

Due to the special choice of our coordinate system, we put

$$\boldsymbol{k} = k_\perp \hat{\boldsymbol{x}} + k_\parallel \hat{\boldsymbol{z}} \tag{5.1}$$

where $\hat{\boldsymbol{x}}$ is a unit vector along the x-axis. The partial cyclotron angular frequency is defined as

$$\Omega_a = q_a B / m_a . \tag{5.2}$$

It should be noted that this definition carries the sign of the charge. $\mathbf{p}_a$ is from (2.4) a symmetric tensor and (3.5) is for each kind of particles equivalent with a set of six equations in the six independent tensor components. These sets of scalar equations can be written with the help of (4.7) as:

$$\omega p_{11}^a - P_\perp^a \boldsymbol{k} \cdot \boldsymbol{v}_a - 2 P_\perp^a k_\perp v_1^a - 2 i \Omega_a p_{12}^a = 0 ,$$

$$\omega p_{22}^a - P_\perp^a \boldsymbol{k} \cdot \boldsymbol{v}_a + 2 i \Omega_a p_{12}^a = 0 ,$$

$$\omega p_{33}^a - P_\parallel^a \boldsymbol{k} \cdot \boldsymbol{v}_a - 2 P_\parallel^a k_\parallel v_3^a = 0 ,$$

$$\omega p_{12}^a - P_\perp^a k_\perp v_2^a + i \Omega_a (p_{11}^a - p_{22}^a) = 0 ,$$

$$\omega p_{13}^a - P_\perp^a k_\perp v_3^a - P_\parallel^a k_\parallel v_1^a - i \Omega_a p_{23}^a + \frac{P_\parallel^a - P_\perp^a}{\alpha c^2} \frac{\Pi_a^2}{\sigma_a} \sum_b \sigma_b (k_\parallel v_1^b - k_\perp v_3^b) = 0 ,$$

$$\omega p_{23}^a - P_\parallel^a k_\parallel v_2^a + i \Omega_a p_{13}^a + \frac{P_\parallel^a - P_\perp^a}{\alpha c^2} \frac{\Pi_a^2}{\sigma_a} k_\parallel \sum_b \sigma_b v_2^b = 0 .$$

After solving this set for the components of the first-order pressure tensor, we find

$$\omega p_{11}^a = 3 P_\perp^a k_\perp v_1^a + P_\perp^a k_\parallel v_3^a + 2 P_\perp^a k_\perp \Omega_a \cdot \frac{2 \Omega_a v_1^a + i \omega v_2^a}{\omega^2 - 4 \Omega_a^2} , \tag{5.3}$$

$$\omega p_{22}^a = P_\perp^a k_\perp v_1^a + P_\perp^a k_\parallel v_3^a - 2 P_\perp^a k_\perp \Omega_a \frac{2 \Omega_a v_1^a + i \omega v_2^a}{\omega^2 - 4 \Omega_a^2} , \tag{5.4}$$

$$\omega p_{33}^a = P_\parallel^a k_\perp v_1^a + 3 P_\parallel^a k_\parallel v_3^a , \tag{5.5}$$

$$p_{12}^a = p_{21}^a = P_\perp^a k_\perp \frac{\omega v_2^a - 2 i \Omega_a v_1^a}{\omega^2 - 4 \Omega_a^2} , \tag{5.6}$$

$$\begin{aligned}(\omega^2 - \Omega_a^2) p_{13}^a = (\omega^2 - \Omega_a^2) p_{31}^a &= \omega P_\perp^a k_\perp v_3^a + \omega P_\parallel^a k_\parallel v_1^a + i \Omega_a P_\parallel^a k_\parallel v_2^a \\ &- \frac{P_\parallel^a - P_\perp^a}{\sigma_a} \Pi_a^2 \sum_b \frac{\sigma_b}{\alpha c^2} (\omega k_\parallel v_1^b + i \Omega_a k_\parallel v_2^b - \omega k_\perp v_3^b) ,\end{aligned} \tag{5.7}$$

$$\begin{aligned}(\omega^2 - \Omega_a^2) p_{23}^a = (\omega^2 - \Omega_a^2) p_{32}^a &= \omega P_\parallel^a k_\parallel v_2^a - i \Omega_a P_\perp^a k_\perp v_3^a - i \Omega_a P_\parallel^a k_\parallel v_1^a \\ &+ \frac{P_\parallel^a - P_\perp^a}{\alpha c^2} \frac{\Pi_a^2}{\sigma_a} \sum_b \sigma_b (i \Omega_a k_\parallel v_1^b - \omega k_\parallel v_2^b - i \Omega_a k_\perp v_3^b) .\end{aligned} \tag{5.8}$$

6. Equations of Motion

The vectorial equations (3.4) can be replaced by the following set of $3N$ equations:

$$\omega^2 v_1^a - (\omega/\varrho_a)(k_\perp p_{11}^a + k_\parallel p_{31}^a) - i \omega (q_a/m_a) e_1 - i \omega \Omega_a v_2^a = 0 ,$$

$$\omega^2 v_2^a - (\omega/\varrho_a)(k_\perp p_{12}^a + k_\parallel p_{32}^a) - i \omega (q_a/m_a) e_2 + i \omega \Omega_a v_1^a = 0 ,$$

$$\omega^2 v_3^a - (\omega/\varrho_a)(k_\perp p_{13}^a + k_\parallel p_{33}^a) - i \omega (q_a/m_a) e_3 = 0 .$$

Substitution of (4.5) and of (5.3—8) transforms this set into

$$\sum_b (A_{ab} v_1^b - i F_{ab} v_2^b + G_{ab} v_3^b) = 0 , \tag{6.1}$$

$$\sum_b (i F_{ab} v_1^b + B_{ab} v_2^b + i H_{ab} v_3^b) = 0 , \tag{6.2}$$

$$\sum_b (G'_{ab} v_1^b - i H'_{ab} v_2^b + C_{ab} v_3^b) = 0 . \tag{6.3}$$

The coefficients in these equations are given by

$$A_{ab} = \left(\omega^2 - 3\,c^2\tau_\perp^a\,k_\perp^2 - \frac{4\,\Omega_a^2 c^2\,\tau_\perp^a\,k_\perp^2}{\omega^2 - 4\,\Omega_a^2} - \frac{\omega^2 c^2\,\tau_\parallel^a k_\parallel^2}{\omega^2 - \Omega_a^2}\right)\sigma_a\,\delta_{ab} + \frac{\Pi_a^2}{\omega^2 - c^2 k^2}\left(c^2 k_\perp^2 - \omega^2 + \frac{\tau_\parallel^a - \tau_\perp^a}{\omega^2 - \Omega_a^2}\,\omega^2 c^2 k_\parallel^2\right)\sigma_b\,, \tag{6.4}$$

$$B_{ab} = \left(1 - \frac{c^2\,\tau_\perp^a\,k_\perp^2}{\omega^2 - 4\,\Omega_a^2} - \frac{c^2\,\tau_\parallel^a k_\parallel^2}{\omega^2 - \Omega_a^2}\right)\omega^2\,\sigma_a\,\delta_{ab} + \frac{\Pi_a^2\,\omega^2}{\omega^2 - c^2 k^2}\left(\frac{\tau_\parallel^a - \tau_\perp^a}{\omega^2 - \Omega_a^2}\,c^2 k_\parallel^2 - 1\right)\sigma_b\,, \tag{6.5}$$

$$C_{ab} = \left(\omega^2 - 3\,c^2\,\tau_\parallel^a k_\parallel^2 - \frac{\omega^2 c^2\,\tau_\perp^a\,k_\perp^2}{\omega^2 - \Omega_a^2}\right)\sigma_a\,\delta_{ab} + \frac{\Pi_a^2}{\omega^2 - c^2 k^2}\left(c^2 k_\parallel^2 - \omega^2 - \frac{\tau_\parallel^a - \tau_\perp^a}{\omega^2 - \Omega_a^2}\,\omega^2 c^2 k_\perp^2\right)\sigma_b\,, \tag{6.6}$$

$$F_{ab} = \left(1 + \frac{2\,c^2\,\tau_\perp^a\,k_\perp^2}{\omega^2 - 4\,\Omega_a^2} + \frac{c^2\,\tau_\parallel^a k_\parallel^2}{\omega^2 - \Omega_a^2}\right)\omega\,\Omega_a\,\sigma_a\,\delta_{ab} - \omega\,\Omega_a\,\frac{\Pi_a^2\,c^2 k_\parallel^2}{\omega^2 - c^2 k^2}\,\frac{\tau_\parallel^a - \tau_\perp^a}{\omega^2 - \Omega_a^2}\,\sigma_b\,, \tag{6.7}$$

$$G_{ab} = -\,c^2 k_\parallel k_\perp\,\tau_\perp^a\left(1 + \frac{\omega^2}{\omega^2 - \Omega_a^2}\right)\sigma_a\,\delta_{ab} + c^2 k_\parallel k_\perp\,\frac{\Pi_a^2}{\omega^2 - c^2 k^2}\left(1 - \frac{\tau_\parallel^a - \tau_\perp^a}{\omega^2 - \Omega_a^2}\,\omega^2\right)\sigma_b\,, \tag{6.8}$$

$$G'_{ab} = -\,c^2 k_\parallel k_\perp\,\tau_\parallel^a\left(1 + \frac{\omega^2}{\omega^2 - \Omega_a^2}\right)\sigma_a\,\delta_{ab} + c^2 k_\parallel k_\perp\,\frac{\Pi_a^2}{\omega^2 - c^2 k^2}\left(1 + \frac{\tau_\parallel^a - \tau_\perp^a}{\omega^2 - \Omega_a^2}\,\omega^2\right)\sigma_b\,, \tag{6.9}$$

$$H_{ab} = \frac{\omega\,\Omega_a\,c^2 k_\parallel k_\perp}{\omega^2 - \Omega_a^2}\left(\tau_\perp^a\,\sigma_a\,\delta_{ab} + \Pi_a^2\,\frac{\tau_\parallel^a - \tau_\perp^a}{\omega^2 - c^2 k^2}\,\sigma_b\right), \tag{6.10}$$

$$H'_{ab} = \frac{\omega\,\Omega_a\,c^2 k_\parallel k_\perp}{\omega^2 - \Omega_a^2}\left(\tau_\parallel^a\,\sigma_a\,\delta_{ab} - \Pi_a^2\,\frac{\tau_\parallel^a - \tau_\perp^a}{\omega^2 - c^2 k^2}\,\sigma_b\right). \tag{6.11}$$

where we have expressed the ratio of the mean thermal velocities to the velocity of light by $\tau_\parallel^a$ or $\tau_\perp^a$, given by

$$\tau_\perp^a = P_\perp^a / c^2\,\varrho_a\,, \quad \tau_\parallel^a = P_\parallel^a / c^2\,\varrho_a\,. \tag{6.12}$$

We now introduce the $N \times N$ matrices

$$\mathbf{A} \equiv (A_{ab})\,,\quad \mathbf{B} \equiv (B_{ab})\,,\quad \mathbf{C} \equiv (C_{ab})\,,$$
$$\mathbf{F} \equiv (F_{ab})\,,\quad \mathbf{G} \equiv (G_{ab})\,,\quad \mathbf{H} \equiv (H_{ab})\,, \tag{6.13}$$
$$\mathbf{G}' \equiv (G'_{ab})\,,\quad \mathbf{H}' \equiv (H'_{ab})$$

and three column matrices with N elements

$$\mathbf{V}_j \equiv (v_j^a) \quad (j = 1, 2, 3). \tag{6.14}$$

The set (6.1—3) can be rewritten as

$$\begin{aligned} \mathbf{A}\cdot\mathbf{V}_1 - i\,\mathbf{F}\cdot\mathbf{V}_2 + \mathbf{G}\cdot\mathbf{V}_3 &= 0\,,\\ i\,\mathbf{F}\cdot\mathbf{V}_1 + \mathbf{B}\cdot\mathbf{V}_2 + i\,\mathbf{H}\cdot\mathbf{V}_3 &= 0\,,\\ \mathbf{G}'\cdot\mathbf{V}_1 - i\,\mathbf{H}'\cdot\mathbf{V}_2 + \mathbf{C}\cdot\mathbf{V}_3 &= 0 \end{aligned}$$

or even shorter as

$$\mathbf{D}\cdot\mathbf{V} = 0 \tag{6.15}$$

if we put

$$\mathbf{D} \equiv \begin{pmatrix} \mathbf{A} & -i\,\mathbf{F} & \mathbf{G} \\ i\,\mathbf{F} & \mathbf{B} & i\,\mathbf{H} \\ \mathbf{G}' & -i\,\mathbf{H}' & \mathbf{C} \end{pmatrix}, \qquad \mathbf{V} \equiv \begin{pmatrix} \mathbf{V}_1 \\ \mathbf{V}_2 \\ \mathbf{V}_3 \end{pmatrix}.$$

In fact $\mathbf{D}$ represents a $3N$ square matrix and $\mathbf{V}$ a column matrix with $3N$ elements for a plasma with N components.

7. Dispersion Relations

If we require a nontrivial solution for (6.15) in the drift velocities, then we have to set

$$\det \mathbf{D} = 0\,. \tag{7.1}$$

As $\mathbf{D}$ is a function of ω and $\boldsymbol{k}$, (7.1) is the desired dispersion relation for the introduced small amplitude plane wave solutions. (7.1) can easily be cast into a more suitable form:

$$\begin{vmatrix} \mathbf{A} & \mathbf{F} & \mathbf{G} \\ \mathbf{F} & \mathbf{B} & \mathbf{H} \\ \mathbf{G}' & \mathbf{H}' & \mathbf{C} \end{vmatrix} = 0\,. \tag{7.2}$$

This determinant is of order $3N$ and has complicated terms. It will in general not be possible to reduce it to a more convenient expression, except by introducing some too intricate mathematical recursion procedure or with the help of supplementary physical approximations. Significant simplifications can occur however, when one considers certain special positions of the wavevector with regard to the external magnetic field. We will restrict the discussion of the dispersion relation in this paper to the aforesaid special cases, where the wavevector is either parallel or perpendicular to the external magnetic induction, without trying to give an exhaustive treatment.

For the exact calculations of some of the dispersion relations in the following sections, we will need the development of determinants of the structure

$$\det(W_{ab}) \equiv \det(X_a Z_a \delta_{ab} - Y_a Z_b) = 0 \quad (7.3)$$

where X_a, Y_a, Z_a will be defined below in each special case separately. Dividing the elements in the bth column by Z_b ($b = 1, \ldots, N$) transform (7.3) into

$$\det(W'_{ab}) \equiv \det(X_a \delta_{ab} - Y_a) = 0 .$$

After division of the elements of the ath row by $-Y_a$, with $a = 1, \ldots, N$, we obtain

$$\det(W''_{ab}) \equiv \det(1 - (X_a/Y_a)\,\delta_{ab}) = 0 .$$

Subtracting the elements of the first column of the corresponding elements of the bth column ($b = 2, \ldots, N$) gives a further reduction to

$$\begin{vmatrix} 1-\frac{X_1}{Y_1} & \frac{X_1}{Y_1} & \cdots & \frac{X_1}{Y_1} & \cdots & \frac{X_1}{Y_1} \\ 1 & -\frac{X_2}{Y_2} & \cdots & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & 0 & \cdots & -\frac{X_a}{Y_a} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & 0 & \cdots & 0 & \cdots & -\frac{X_N}{Y_N} \end{vmatrix} = 0 .$$

The development is now straightforward and results in

$$\prod_{a=1}^{N} X_a/Y_a = \sum_{a=1}^{N} U_a \quad (7.4)$$

if we put

$$U_a = \prod_{\substack{b=1 \\ b \neq a}}^{N} X_b/Y_b .$$

After division of (7.4) by its left hand side, the dispersion formula becomes

$$1 = \sum_a Y_a/X_a . \quad (7.5)$$

8. A Parallel Magnetic Induction

We simply mean by a parallel magnetic induction that in this case wavevector and external magnetic induction are parallel. Putting hence $k_\perp = 0$ and $k_\parallel = k$ gives in (6.4—11):

$$A_{ab} = B_{ab} = \omega^2 \left(1 - \frac{\tau_\parallel^a c^2 k^2}{\omega^2 - \Omega_a^2}\right) \sigma_a \delta_{ab} - \frac{\omega^2 \Pi_a^2}{\omega^2 - c^2 k^2} \left(1 - \frac{\tau_\parallel^a - \tau_\perp^a}{\omega^2 - \Omega_a^2} c^2 k^2\right) \sigma_b ,$$

$$C_{ab} = (\omega^2 - 3\,\tau_\parallel^a c^2 k^2)\,\sigma_a \delta_{ab} - \Pi_a^2 \sigma_b ,$$

$$F_{ab} = \omega\,\Omega_a \left(1 + \frac{\tau_\parallel^a c^2 k^2}{\omega^2 - \Omega_a^2}\right) \sigma_a \delta_{ab} - \omega\,\Omega_a \frac{c^2 k^2 \Pi_a^2}{\omega^2 - c^2 k^2} \frac{\tau_\parallel^a - \tau_\perp^a}{\omega^2 - \Omega_a^2} \sigma_b ,$$

$$G_{ab} = G'_{ab} = H_{ab} = H'_{ab} = 0 .$$

The dispersion determinant is now of the form

$$\begin{vmatrix} \mathbf{A} & \mathbf{F} & \mathbf{O} \\ \mathbf{F} & \mathbf{A} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & \mathbf{C} \end{vmatrix} = 0$$

and falls apart into

$$\begin{vmatrix} \mathbf{A} & \mathbf{F} \\ \mathbf{F} & \mathbf{A} \end{vmatrix} = 0 \quad (8.1)$$

and into

$$\det \mathbf{C} = 0 . \quad (8.2)$$

In the first case we clearly are dealing with transverse waves as for all a we find v_3^a to vanish, as it is in general not possible to find values of ω and $\boldsymbol{k}$ which fulfil both (8.1) and (8.2). In the other case we have the longitudinal waves as now every v_1^a and v_2^a disappears.

(8.1) can successively be transformed into

$$\begin{vmatrix} \mathbf{A}+\mathbf{F} & \mathbf{F}+\mathbf{A} \\ \mathbf{F} & \mathbf{A} \end{vmatrix} = 0$$

and into

$$\begin{vmatrix} \mathbf{A}+\mathbf{F} & \mathbf{O} \\ \mathbf{F} & \mathbf{A}-\mathbf{F} \end{vmatrix} = 0$$

basing us on the rules for matrix addition and subtraction. The last equation is equivalent to

$$\det(\mathbf{A} \pm \mathbf{F}) = 0 \quad (8.3)$$

with elements

$$A_{ab} \pm F_{ab} = \left(\omega \pm \Omega_a - \frac{\tau_\parallel^a c^2 k^2}{\omega \pm \Omega_a}\right) \omega\,\sigma_a \delta_{ab} - \frac{\Pi_a^2}{\omega^2 - c^2 k^2} \left(\omega - \frac{\tau_\parallel^a - \tau_\perp^a}{\omega \pm \Omega_a} c^2 k^2\right) \omega\,\sigma_b .$$

As (8.3) clearly possesses the structure of (7.3), with

$$X_a = \omega \pm \Omega_a - \frac{\tau_\parallel^a c^2 k^2}{\omega \pm \Omega_a} ,$$

$$Y_a = \frac{\Pi_a^2}{\omega^2 - c^2 k^2} \left(\omega - \frac{\tau_\parallel^a - \tau_\perp^a}{\omega \pm \Omega_a} c^2 k^2\right) ,$$

$$Z_a = \omega\,\sigma_a$$

we find from (7.5) the *dispersion formula for the transverse waves* in the presence of a parallel magnetic induction

$$1 = \sum_a \frac{[\Pi_a^2/(\omega^2 - c^2 k^2)](\omega - [(\tau_\parallel^a - \tau_\perp^a)/(\omega \pm \Omega_a)]\, c^2 k^2)}{\omega \pm \Omega_a - \tau_\parallel^a c^2 k^2/(\omega \pm \Omega_a)}$$

or

$$\omega^2 = c^2 k^2 + \sum_a \frac{\omega - [(\tau_\parallel^a - \tau_\perp^a)/(\omega \pm \Omega_a)]\, c^2 k^2}{\omega \pm \Omega_a - \tau_\parallel^a c^2 k^2/(\omega \pm \Omega_a)} \Pi_a^2 . \qquad (8.4)$$

If we restrict ourselves to a two-component plasma, we can transform (8.4) into the corresponding formula of JAGGI[7]. In his paper, he claims that the generalization of his two-component formula to an N-component treatment is obvious, but he does not give any proof of his statement. On the contrary, the way in which he obtained his dispersion equations is not easily suitable for extension. In our derivation the dispersion relation (8.4) not only is proved, but also written in a more handsome form.

If we had started our analysis with an isotropic equilibrium pressure for each component of the plasma, we would have obtained the dispersion equation

$$\omega^2 = c^2 k^2 + \sum_a \frac{\omega \Pi_a^2}{\omega \pm \Omega_a - \tau_a c^2 k^2/(\omega \pm \Omega_a)} . \qquad (8.5)$$

Applied to an electron plasma, this result is consistent with one of the formulas given by PYTTE-BLANKEN[8] in the absence of collisions.

With the supplementary hypothesis that for every a

$$\tau_a \ll (\omega \pm \Omega_a)^2/c^2 k^2$$

we can rewrite (8.5) as

$$\omega^2 = c^2 k^2 + \sum_a \frac{(\omega \pm \Omega_a)^2 + \tau_a c^2 k^2}{(\omega \pm \Omega_a)^3} \omega \Pi_a^2$$

reproducing a result found by QUÉMADA[9].

For the *longitudinal waves* we have to express (8.2), which also exhibits the structure of (7.3) with

$$X_a = \omega^2 - 3\, \tau_\parallel^a c^2 k^2 ,$$
$$Y_a = \Pi_a^2 ,$$
$$Z_a = \sigma_a$$

from the expression found for C_{ab}. The dispersion relation hence follows from (7.5) as

$$1 = \sum_a \Pi_a^2/(\omega^2 - 3\, \tau_\parallel^a c^2 k^2) . \qquad (8.6)$$

Our results were derived under the assumption of a low temperature plasma, written explicitly as

$$\tau_{\parallel, \perp}^a \ll \omega^2/c^2 k^2$$

so that (8.6) is reshaped to

$$\omega^2 = \sum_a \Pi_a^2 (1 + 3 (c^2 k^2/\omega^2)\, \tau_\parallel^a) . \qquad (8.7)$$

A comparison of this equation with the analogon obtained in the separate adiabatic approximation[10] shows that we have to put every Γ_a equal to 3 to gain agreement.

9. A Perpendicular Magnetic Induction

The other propagation modes that yield simple dispersion relations are obtained when the wavevector and the external magnetic induction stand perpendicular to each other. This means that for these other principal waves we have to put $k_\parallel$ equal to zero in (6.4—11). With $k_\perp$ written as k we find

$$A_{ab} = \left(\omega^2 - 3\, \tau_\perp^a c^2 k^2 - \frac{4\, \Omega_a^2 \tau_\perp^a c^2 k^2}{\omega^2 - 4\, \Omega_a^2}\right) \sigma_a \delta_{ab} - \Pi_a^2 \sigma_b ,$$

$$B_{ab} = \omega^2 \left(1 - \frac{\tau_\perp^a c^2 k^2}{\omega^2 - 4\, \Omega_a^2}\right) \sigma_a \delta_{ab} - \frac{\omega^2 \Pi_a^2}{\omega^2 - c^2 k^2} \sigma_b ,$$

$$C_{ab} = \omega^2 \left(1 - \frac{\tau_\perp^a c^2 k^2}{\omega^2 - \Omega_a{}^2}\right) \sigma_a \delta_{ab} - \frac{\omega^2 \Pi_a^2}{\omega^2 - c^2 k^2} \left(1 + \frac{\tau_\parallel^a - \tau_\perp^a}{\omega^2 - \Omega_a^2} c^2 k^2\right) \sigma_b ,$$

$$F_{ab} = \omega\, \Omega_a \left(1 + \frac{2\, \tau_\perp^a c^2 k^2}{\omega^2 - 4\, \Omega_a^2}\right) \sigma_a \delta_{ab} , \qquad G_{ab} = G'_{ab} = H_{ab} = H'_{ab} = 0 .$$

[7] R. K. JAGGI, Phys. Fluids 5, 949 [1962].
[8] A. PYTTE and R. BLANKEN, Phys. Rev. 133, A 668 [1964].
[9] D. QUÉMADA, La propagation des ondes dans les plasmas, cours d'été de physique spatiale, polyc., Toulouse 1966.
[10] F. G. VERHEEST, Simon Stevin 40, 36 [1966].

The determinant (7.2) is here reduced to

$$\begin{vmatrix} \mathbf{A} & \mathbf{F} & \mathbf{O} \\ \mathbf{F} & \mathbf{B} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & \mathbf{C} \end{vmatrix} = 0 .$$

The wave with dispersion relation

$$\det \mathbf{C} = 0 \tag{9.1}$$

travels along the external magnetic induction, every v_1^a and v_2^a vanishing. We furthermore learn from (4.5) that $\boldsymbol{e}$ is parallel to $\boldsymbol{B}$, hence (9.1) refers to the well-known mode with $\boldsymbol{e}$ along $\boldsymbol{B}$. The found expression here C_{ab} shows us that we have to set

$$X_a = 1 - \tau_\perp^a \, c^2 k^2/(\omega^2 - \Omega_a^2),$$
$$Y_a = [\Pi_a^2/(\omega^2 - c^2 k^2)] \, (1 + [(\tau_\parallel^a - \tau_\perp^a)/(\omega^2 - \Omega_a^2)] \, c^2 k^2),$$
$$Z_a = \omega^2 \sigma_a$$

in (7.5) to get as a dispersion formula

$$1 = \sum_a \frac{[\Pi_a^2/(\omega^2 - c^2 k^2)] \, (1 + [(\tau_\parallel^a - \tau_\perp^a)/(\omega^2 - \Omega_a^2)] \, c^2 k^2)}{1 - \tau_\perp^a \, c^2 k^2/(\omega^2 - \Omega_a^2)}$$

or also

$$\omega^2 = c^2 k^2 + \sum_a \frac{\omega^2 - \Omega_a^2 + c^2 k^2 (\tau_\parallel^a - \tau_\perp^a)}{\omega^2 - \Omega_a^2 - \tau_\perp^a \, c^2 k^2} \Pi_a^2 . \tag{9.2}$$

Here also we achieved a proved generalization of one of JAGGI's formulas[7]. If we work with scalar equilibrium pressures, as was done by QUÉMADA[9], (9.2) reduces to

$$\omega^2 = c^2 k^2 + \sum_a \frac{\Pi_a^2}{1 - \tau_a \, c^2 k^2/(\omega^2 - \Omega_a^2)} . \tag{9.3}$$

Writing (9.3) once again for an electron plasma ($N = 1$) yields one of the dispersion formulas obtained by PYTTE-BLANKEN[8] in the collisionless limit.

The remainder of the dispersion equation is

$$\begin{vmatrix} \mathbf{A} & \mathbf{F} \\ \mathbf{F} & \mathbf{B} \end{vmatrix} = 0 .$$

After a long and tedious but rather straightforward development of this determinant we get

$$c^2 k^2 \left(1 - \sum_a \frac{\psi_a \Pi_a^2}{\omega^2 \varphi_a \psi_a - \Omega_a^2 \chi_a^2}\right) = \omega^2 \left(1 - \sum_a \frac{\varphi_a \Pi_a^2}{\omega^2 \varphi_a \psi_a - \Omega_a^2 \chi_a^2}\right)\left(1 - \sum_a \frac{\psi_a \Pi_a^2}{\omega^2 \varphi_a \psi_a - \Omega_a^2 \chi_a^2}\right) - \left(\sum_a \frac{\chi_a \Omega_a \Pi_a^2}{\omega^2 \varphi_a \psi_a - \Omega_a^2 \chi_a^2}\right)^2 \tag{9.4}$$

where we have written

$$\varphi_a = 1 - [3 \tau_\perp^a - 4 \Omega_a^2 \tau_\perp^a/(\omega^2 - 4 \Omega_a^2)] \cdot c^2 k^2/\omega^2 ,$$
$$\psi_a = 1 - \tau_\perp^a \, c^2 k^2/(\omega^2 - 4 \Omega_a^2), \quad \chi_a = 1 + 2 \tau_\perp^a \, c^2 k^2/(\omega^2 - 4 \Omega_a^2).$$

This formula (9.4) is the generalization for a warm plasma of the cold plasma dispersion relation for the transverse extraordinary mode, as given by STIX[3] and SMITH-BRICE[11].

10. Conclusions

Working with the set of moment equations, even if derived from a BOLTZMANN-VLASOV equation, requires an additional hypothesis to close the system. We adopted here a low-temperature or complete adiabaticity model, neglecting the heat flow gradient in every pressure tensor equation. The assumption of plane wave phenomena as small perturbations was then sufficient to eliminate all first-order variables, but for the components of the drift velocities.

As the set of equations in these velocity components was homogeneous, we immediately found

[11] R. L. SMITH and N. BRICE, J. Geophys. Res. **69**, 5029 [1964].

a general dispersion relation. Worthwhile to investigate it as it was, we felt it however not feasable without additional physical approximations or impossibly difficult mathematical recursion procedures to give more explicit formulas in the most general case.

For the principal waves however, propagating along or across the external magnetic induction, some short formulas were found. A discussion of these results compared with existing work in the field showed the inclusion of many known formulas as special cases of our relations. The present form of these relations makes them easily suitable for various other physical approximations, which can be as varied as the existing literature on plasma waves. This, however, is beyond the scope of this article.

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Multicomponent Beam-Plasma Waves

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The linearization procedure is applied to the equations governing a beam-plasma system, in which the stream velocities and the wavevector are parallel to the external magnetic induction. No special constraints are imposed on the parameters characterizing the constituent fluids in the equilibrium state of this macroscopic picture. From the MAXWELL equations an expression for the electromagnetic field of the wave is obtained and substituted in the equations of motion. The components of the first-order pressure tensors are computed in the low-temperature approximation, but without recurring to the strong magnetic induction CGL hypothesis. Since the equations of motion are now expressed only in the components of the perturbations of the drift velocities, the dispersion relations follow immediately. These relations are applicable to all beam-plasma systems comprised between the now conventional multicomponent plasma and the system of beams of charged particles. Some known cold beam-plasma cases are included in the general dispersion equations.

1. Introduction

In our investigation of multicomponent beam-plasma waves, we will adhere as closely as possible to the treatment of waves in multicomponent VLASOV plasmas without zero-order drift velocities[1] (henceforth referred to as I). We will in particular make use of the same notations and formulas of I if possible, and refer to them as to (I ...).

The set of basic equations (I.2.6—8) will be linearized here in a similar fashion as in I, with the help however of the following additional hypotheses.

(1) In our macroscopic picture of the plasma a zero-order drift velocity is introduced for every constituent fluid. We now must designate by „component of the beam-plasma system" the collection of all particles which have the same values for the complete set of characterizing zero-order parameters, such as charge, mass, density, equilibrium drift velocity and equilibrium pressure. Electrons with different beam velocities hence belong to different components of the system. Furthermore, at any stage of the analysis some of these finite drift velocities can be put equal to zero to describe the pure plasma part of the system. The results thus will be applicable to streaming multicomponent plasmas, beams of charged particles and every other combination of these plasmas and beams.

(2) The most striking features of the introduction of finite drift velocities are noted in the direction of the external magnetic induction, if present, and of the wavevector. We therefore restrict ourselves in this study to wave propagation parallel to the external magnetic induction. For mathematical simplicity we direct the finite drift velocities along the now privileged z-axis:

$$\boldsymbol{V}_a = V_a \hat{\boldsymbol{z}}, \quad \boldsymbol{B} = B \hat{\boldsymbol{z}}, \quad \boldsymbol{k} = k \hat{\boldsymbol{z}} \qquad (1.1)$$

(3) A scalar equilibrium pressure is adopted to avoid too intricate formulas, changing (I.3.2) into

$$\mathbf{P}_a = P_a \, \mathbf{I} \qquad (1.2)$$

for every $a = 1, \ldots, N$. Our treatment, however, still caters for anisotropic pressure variations. As we do

[1] F. G. VERHEEST, Z. Naturforschg. 22a, 1927 [1967].